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On the Values of the Constants in the Equation ${}_rA_r x^{(r)} + {}_rA_{r-1} x^{(r-1)} + \ldots + {}_rA_t x^{(t)} + \ldots + {}_rA_0 - y_x = 0$; *obtained by the method of least squares, from the* $n+1$ *values of* y_x *when* $x = 0, 1, 2, \ldots n$; *n being greater than* r. By C. Carpmael, Esq.

Suppose it known, or assumed, that the value of a quantity y_x (the approximate value of which is known when $x = 0$, and when $x = 1, 2, \&c. \ldots n$) may be expressed as the sum of a series of factorials of the form ${}_rA_t x^{(t)}$, where ${}_rA_t$ is the coefficient of $x^{(t)}$ when the highest factorial in the series is $x^{(r)}$.

The object of the following investigation is to determine the values of the coefficients, ${}_rA_r$ &c., which make the sum of the squares of the differences between the given values of y and those obtained from the formula a minimum.*

Now in order that

$$\Sigma_{x=0}^{x=n} ({}_rA_r x^{(r)} + {}_rA_{r-1} x^{(r-1)} + \ldots + {}_rA_t x^{(t)} + \ldots + {}_rA_0 - y_x)^2$$

may be a minimum, ${}_rA_r$ &c. must satisfy simultaneously the $\overline{r+1}$ equations of the form

$${}_rA_r \Sigma_{x=0}^{x=n} x^{(r)} x^{(s)} + {}_rA_{r-1} \Sigma_{x=0}^{x=n} x^{(r-1)} x^{(s)} + \ldots + {}_rA_t \Sigma_{x=0}^{x=n} x^{(t)} x^{(s)} + \ldots$$
$$+ {}_rA_0 \Sigma_{x=0}^{x=n} x^{(s)} - \Sigma_{x=0}^{x=n} x^{(s)} y_x = 0,$$

s having in succession the values $r, r-1, r-2, \ldots, 0$.

The value of ${}_rA_r$ obtained from these equations is a fraction, whose denominator is a determinant of the $\overline{r+1}$th order, having

$$\Sigma_{x=0}^{x=n} x^{(s)} x^{(t)}$$

* The notation is: $x^{(r)} = x\ (x-1) \ldots (x-r+1)$; $\underline{|r} = 1.2.3 \ldots r$; $\Sigma_{x=0}^{x=n} y_x = y_n + y_{n-1} \ldots + y_0$.—ED.

for the constituent in its $\overline{r-s+1}$ th row and $\overline{r-t+1}$ th column; and whose numerator may be obtained from the denominator by substituting constituents of the form

$$\Sigma_{x=0}^{x=n} x^{(s)} y_x$$

for those of the form

$$\Sigma_{x=0}^{x=n} x^{(s)} x^{(r)}$$

in the first column.

Performing the summations wherever y is not involved, by aid of the formula

$$\Delta^{-1} u_x v_x = u_x \Delta^{-1} v_x - \Delta u_x \Delta^{-2} v_{x+1} + \Delta^2 u_x \Delta^{-3} v_{x+2} - \&c.,$$

we obtain for the value of the constituent in the $\overline{r-s+1}$ th row and $\overline{r-t+1}$ th column

$$\overline{n+1}^{(t)} \frac{\overline{n+1}^{(s+1)}}{s+1} - t\, \overline{n+1}^{(t-1)} \frac{\overline{n+2}^{(s+2)}}{\overline{s+1}\;\overline{s+2}} + \ldots + (-1)^t \frac{|\underline{t}\;|\underline{s}}{|\underline{s+t+1}}\, \overline{n+t+1}^{(s+t+1)}. \qquad \text{(i)}$$

In the last column $t = 0$, and (i) reduces to

$$\frac{\overline{n+1}^{(s+1)}}{s+1};$$

in the last but one $t = 1$, and (i) becomes

$$\overline{n+1}\, \frac{\overline{n+1}^{(s+1)}}{s+1} - \frac{\overline{n+2}^{(s+2)}}{\overline{s+1}\;\overline{s+2}};$$

and so on. Subtracting $\overline{n+1}$ times the last column from the last but one, we obtain new determinants of the same values as the previous ones, and having the constituents in the last column but one of the form

$$-\frac{\overline{n+2}^{(s+2)}}{\overline{s+1}\;\overline{s+2}}.$$

Similarly, by subtracting $2 . \overline{n+1}$ times this new column, and $\overline{n+1} . n$ times the last column from the last column but two, we reduce its constituents to the form

$$2\, \frac{\overline{n+3}^{(s+3)}}{\overline{s+1}\;\overline{s+2}\;\overline{s+3}}.$$

By proceeding in this way we reduce the determinants, without

changing their values, to others having the constituents in the $\overline{r-s+1}$ th row and $\overline{r-t+1}$ th column of the form

$$(-1)^t \frac{\underline{|t}\;\underline{|s}}{\underline{|s+t+1}}\;\overline{n+t+1}^{(s+t+1)},$$

s and t having any value from 0 to r, except in the first column of the numerator, which is left as before.

Now divide the $\overline{r-s+1}$ th row in each of these new determinants by

$$\underline{|s}\;\overline{n+1}^{(s+1)},$$

and the $\overline{r-t+1}$ th column by

$$(-1)^t \frac{\underline{|t}\;\overline{n+t+1}^{(t)}}{\underline{|r+t+1}}.$$

This will reduce the constituent in the $\overline{r-s+1}$ th row and the $\overline{r-t+1}$ th column to the form

$$\frac{\underline{|r+t+1}}{\underline{|s+t+1}} \quad \text{or} \quad \overline{r+t+1}^{(r-s)},$$

except in the first column of the numerator, where the $\overline{r-s+1}$ th constituent will be

$$(-1)^r \frac{\underline{|2r+1}}{\underline{|r}\;\underline{|s}}\;\frac{\Sigma_{x=o}^{x=n}\, x^{(s)}\, y_x}{\overline{n+r+1}^{(r+s+1)}}.$$

As we here divide both the determinants by the same quantities, the value of their ratio is unchanged.

The value of ${}_rA_r$ thus obtained may be expanded in the form

$${}_rA_r = \Sigma_{s=0}^{s=r}\,(-1)^s \frac{\underline{|2r+1}}{\underline{|r}\;\underline{|s}}\;\frac{1}{\overline{n+r+1}^{(r+s+1)}}\;{}_rR_s\;\Sigma_{x=o}^{x=n}\, x^{(s)}\, y_x, \qquad \text{(ii)}$$

where ${}_rR_s$ is of the form

$$\frac{\begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 2r & 2r-1 & \cdots & r+t+1 & \cdots & r+1 \\ (2r)^{(2)} & (2r-1)^{(2)} & \cdots & (r+t+1)^{(2)} & \cdots & (r+1)^{(2)} \\ \vdots & \vdots & & \vdots & & \vdots \\ (2r)^{(r-s-1)} & (2r-1)^{(r-s-1)} & \cdots & (r+t+1)^{(r-s-1)} & \cdots & (r+1)^{(r-s-1)} \\ (2r)^{(r-s+1)} & (2r-1)^{(r-s+1)} & \cdots & (r+t+1)^{(r-s+1)} & \cdots & (r+1)^{(r-s+1)} \\ \vdots & \vdots & & \vdots & & \vdots \\ (2r)^{(r)} & (2r-1)^{(r)} & \cdots & (r+t+1)^{(r)} & \cdots & (r+1)^{(r)} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2r+1 & 2r & \cdots & r+1 \\ (2r+1)^{(2)} & 2r^{(2)} & \cdots & \overline{r+1}^{(2)} \\ \vdots & \vdots & & \vdots \\ (2r+1)^{(r)} & (2r)^{(r)} & \cdots & \overline{r+1}^{(r)} \end{vmatrix}}$$

the numerator being the same as the denominator, except that the first column and $\overline{r-s+1}$ th row are wanting.

Now reverse the order of the columns, to do which will require $\frac{1}{2}r\ \overline{r-1}$ interchanges of adjacent columns in the numerator and $\frac{1}{2}r\ \overline{r+1}$ in the denominator, and will therefore alter the sign of the fraction, or not, according as $(-1)^r$ is positive or negative.

Hence

$$
{}_rR_s = (-1)^r \frac{\begin{vmatrix} 1 & \cdots & 1 \\ r+1 & \cdots & 2r \\ \vdots & & \vdots \\ (r+1)^{(r-s-1)} & \cdots & (2r)^{(r-s-1)} \\ (r+1)^{(r-s+1)} & \cdots & (2r)^{(r-s+1)} \\ \vdots & & \vdots \\ (r+1)^{(r)} & \cdots & (2r)^{(r)} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ r+1 & \cdots & r+t+1 & \cdots & 2r+1 \\ \vdots & & \vdots & & \vdots \\ (\overline{r+1})^{(r-s)} & \cdots & (\overline{r+t+1})^{(r-s)} & \cdots & (2r+1)^{(r-s)} \\ \vdots & & \vdots & & \vdots \\ (\overline{r+1})^{(r)} & \cdots & (r+t+1)^{(r)} & \cdots & (2r+1)^{(r)} \end{vmatrix}} \quad . \quad \text{(iii)}
$$

In the determinants in (iii) from each column, beginning at the right, subtract that next to its left; the determinants will thus be each reduced one order lower, for the constituents of the first row are all reduced to zero, except the first, which remains unity. Then divide each row by the factor common to that row. These common factors are the same in both numerator and denominator, except that one factor $\overline{r-s}$ occurs in the denominator and not in the numerator. The value of ${}_rR_s$ is now reduced to the form

$$\frac{(-1)^r}{r-s}\,\frac{\begin{pmatrix} 1 & \cdots\cdots & 1 \\ r+1 & \cdots\cdots & 2r-1 \\ \vdots & & \vdots \\ (r+1)^{(r-s-2)} & \cdots\cdots & (2r-1)^{(r-s-2)} \\ (r+1)^{(r-s)} & \cdots\cdots & (2r-1)^{(r-s)} \\ \vdots & & \vdots \\ (r+1)^{(r-1)} & \cdots\cdots & (2r-1)^{(r-1)} \end{pmatrix}}{\begin{pmatrix} 1 & \cdots\cdots & 1 \\ r+1 & \cdots\cdots & 2r \\ \vdots & & \vdots \\ (r+1)^{(r-s-1)} & \cdots\cdots & (2r)^{(r-s-1)} \\ \vdots & & \vdots \\ (r+1)^{(r-1)} & \cdots\cdots & (2r)^{(r-1)} \end{pmatrix}}$$

where the determinants are similar in form to those in (iii) but one order lower, and the missing row in the numerator is now that containing factorials of $\overline{r-s-1}$ factors.

Repeat the reduction in this way $\overline{r-s}$ times, including that already performed, then the value of ${}_rR_s$ will be reduced to the form

$$\frac{(-1)^r}{\underline{|r-s}}\,\frac{\begin{pmatrix} r+1 & r+2 & \cdots\cdots & r+s \\ (r+1)^{(2)} & (r+2)^{(2)} & \cdots\cdots & (r+s)^{(2)} \\ \vdots & \vdots & & \vdots \\ (r+1)^{(s)} & (r+2)^{(s)} & \cdots\cdots & (r+s)^{(s)} \end{pmatrix}}{\begin{pmatrix} 1 & 1 & \cdots\cdots & 1 \\ r+1 & r+2 & \cdots\cdots & r+s+1 \\ (r+1)^{(2)} & (r+2)^{(2)} & \cdots\cdots & (r+s+1)^{(2)} \\ \vdots & \vdots & & \vdots \\ (r+1)^{(s)} & (r+2)^{(s)} & \cdots\cdots & (r+s+1)^{(s)} \end{pmatrix}}$$

The determinants have now been reduced $\overline{r-s}$ orders lower than at first, and the row of the denominator which is missing in the numerator is that in which every constituent is unity.

Taking out the factors in the numerator which occur throughout any column, we get

$$
{}_r\mathrm{R}_s = (-1)^r \frac{\underline{|r+s}}{\underline{|r}\ \underline{|r-s}} \cdot \frac{\begin{pmatrix} 1 & \cdots & 1 \\ r & \cdots & r+s-1 \\ \vdots & & \vdots \\ r^{(s-1)} & \cdots & (r+s-1)^{(s-1)} \end{pmatrix}}{\begin{pmatrix} 1 & \cdots & 1 \\ r+1 & \cdots & r+s+1 \\ \vdots & & \vdots \\ (r+1)^{(s)} & \cdots & (r+s+1)^{(s)} \end{pmatrix}}
$$

The row now missing in the numerator is the last, and in the numerator r occurs instead of $r+1$.

Reduce the order of the determinants in the same way as before, s times, we get

$$
{}_r\mathrm{R}_s = (-1)^r \frac{\underline{|r+s}}{\underline{|r}\ \underline{|s}\ \underline{|r-s}}.
$$

Substituting this value for ${}_r\mathrm{R}_s$ in (ii), we get for the value of ${}_r\mathrm{A}_r$

$$
\mathrm{A}_r = \Sigma_{s=0}^{s=r} (-1)^{r+s} \frac{\underline{|2r+1}}{(\underline{|r}\ \underline{|s})^2} \frac{\underline{|r+s}}{\underline{|r-s}} \frac{1}{(n+r+1)^{(r+s+1)}} \Sigma_{x=0}^{x=n} x^{(s)} y_x. \qquad \text{(A)}
$$

From this formula we can without much labour determine the coefficient, in any numerical example, of the highest factorial in the expression for y.

The values of the remaining coefficients are most easily obtained by an indirect method, as follows :—

Let ${}_r\mathrm{B}_\rho$ be the value of ${}_r\mathrm{A}_r$ when $x^{(\rho)}$ is substituted for y_x, then

$$
{}_r\mathrm{B}_\rho = \Sigma_{s=0}^{s=r} (-1)^{r+s} \frac{\underline{|2r+1}}{(\underline{|r}\ \underline{|s})^2} \frac{\underline{|r+s}}{\underline{|r-s}} \frac{1}{(n+r+1)^{(r+s+1)}} \Sigma_{x=0}^{x=n} x^{(s)} x^{(\rho)}.
$$

Now by aid of the formula

$$
\Delta^{-1} u_x v_x = u_{x-1} \Delta^{-1} v_x - \Delta u_{x-2} \Delta^{-2} v_x + \&c.,
$$

we have

$$\Sigma_{x=0}^{x=n} x^{(s)} x^{(\rho)} = n^{(s)} \frac{(n+1)^{(\rho+1)}}{\rho+1} - s\,(n-1)^{(s-1)} \frac{(n+1)^{(\rho+2)}}{(\rho+1)(\rho+2)} + \ldots$$
$$+ (-1)^p \frac{|\underline{s}}{|\underline{s-p}} (n-p)^{(s-p)} \frac{|\underline{\rho}}{|\underline{\rho+p+1}} (n+1)^{(\rho+p+1)} + \&c.$$

Hence ${}_r\mathrm{B}_\rho$ may be written in the form

$$\Sigma_{s=0}^{s=r} \frac{|\underline{2r+1}}{(|\underline{r}\;|\underline{s})^2} \frac{|\underline{r+s}}{|\underline{r-s}} \frac{1}{(n+r+1)^{(r+s+1)}} \Sigma_{p=0}^{p=s} \frac{|\underline{s}}{|\underline{s-p}} \frac{|\underline{n-p}}{|\underline{n-s}} \frac{|\underline{\rho}}{|\underline{\rho+p+1}} (n+1)^{(\rho+p+1)} (-1)^{p+r+s};$$

or

$$\Sigma_{s=0}^{s=r} \Sigma_{p=0}^{p=s} (-1)^{p+r+s} \frac{|\underline{2r+1}}{(|\underline{r})^2\;|\underline{n+r+1}} \frac{|\underline{r+s}}{|\underline{s}\;|\underline{r-s}\;|\underline{s-p}} \frac{|\underline{n-p}\;|\underline{\rho}\;|\underline{n+1}}{|\underline{\rho+p+1}\;|\underline{n-\rho-p}};$$

or, changing the order of the summations,

$${}_r\mathrm{B}_\rho = \Sigma_{p=0}^{p=r} \Sigma_{s=p}^{s=r} (-1)^{p+r+s} \frac{|\underline{2r+1}}{(|\underline{r})^2\;|\underline{n+r+1}} \frac{|\underline{n-p}\;|\underline{\rho}\;|\underline{n+1}}{|\underline{\rho+p+1}\;|\underline{n-\rho-p}} \frac{|\underline{r+s}}{|\underline{s}\;|\underline{r-s}\;|\underline{s-p}}.$$

Now

$$\frac{|\underline{r+s}}{|\underline{s}} \text{ is } = \text{coeff. of } x^s \text{ in expansion of } |\underline{r}\,(1-x)^{-r-1},$$

and

$$(-1)^{r\pm s} \frac{1}{|\underline{r-s}\;|\underline{s-p}} \text{ is } = \text{coeff. of } x^{r-s} \text{ in expansion of } \frac{(1-x)^{r-p}}{|\underline{r-p}},$$

when s has any value from $s=p$ to $s=r$; therefore

$$\Sigma_{s=p}^{s=r} (-1)^{r+s} \frac{|\underline{r+s}}{|\underline{s}} \frac{1}{|\underline{r-s}\;|\underline{s-p}}$$

is $=$ coeff. of x^r in expansion of

$$\frac{|\underline{r}}{|\underline{r-p}} (1-x)^{-p-1},$$

that is, it is $=$

$$\frac{|\underline{r}}{|\underline{r-p}} \frac{|\underline{r+p}}{|\underline{r}\;|\underline{p}} \quad \text{or} \quad \frac{|\underline{r+p}}{|\underline{r-p}\;|\underline{p}}.$$

Hence

$${}_r\mathrm{B}_\rho = \Sigma_{p=0}^{p=r} (-1)^p \frac{|\underline{2r+1}\;|\underline{\rho}\;|\underline{n+1}}{(|\underline{r})^2\;|\underline{n+r+1}} \frac{|\underline{n-p}\;|\underline{r+p}}{|\underline{\rho+p+1}\;|\underline{n-\rho-p}\;|\underline{r-p}\;|\underline{p}},$$

which may be written

$${}_r\mathrm{B}_\rho = \frac{|\underline{2r+1}\;|\underline{n+1}\;|\underline{\rho}}{(|\underline{r})^3\;|\underline{n+r+1}} \left(1-\frac{\mathrm{E}'}{\mathrm{E}}\right)^r n^{(\rho)} (r+1)^{(-\rho+r-1)},$$

where

E' applies only to $(r+1)^{(-\rho+r-1)}$ and $E'\,(r+1)^{(-\rho+r-1)} = (r+2)^{(-\rho+r-1)}$,

and

E applies only to $n^{(\rho)}$ and $E\,n^{(\rho)} = (n+1)^{(\rho)}$.

Hence

$$_rB_\rho = \frac{\underline{|2r+1}\;\underline{|n+1}\;\underline{|\rho}}{(\underline{|r})^3\;\underline{|n+r+1}}\left(\frac{\Delta-\Delta'}{E}\right)^r n^{(\rho)}\,(r+1)^{(-\rho+r-1)},$$

$$= \frac{\underline{|2r+1}\;\underline{|n+1}\;\underline{|\rho}}{(\underline{|r})^3\;\underline{|n+r+1}}\,\Sigma_{p=0}^{p=r}\frac{\underline{|r}\,(-1)^{r-p}}{\underline{|p}\;\underline{|r-p}}\,\Delta^p\,(n-r)^{(\rho)}\,\Delta'^{r-p}\,(r+1)^{(-\rho+r-1)},$$

$$= \frac{\underline{|2r+1}\;\underline{|n+1}\;\underline{|\rho}}{(\underline{|r})^3\;\underline{|n+r+1}}\,\Sigma_{p=0}^{p=r}\frac{\underline{|r}}{\underline{|p}\;\underline{|r-p}}\,\rho^{(p)}\,(n-r)^{(\rho-p)}\,\frac{\underline{|\rho-p}}{\underline{|\rho-r}}\,(r+1)^{(-\rho+p-1)},$$

$$= \frac{\underline{|2r+1}\;\underline{|n-r}}{\underline{|n+r+1}\;\underline{|\rho-r}}\left(\frac{\underline{|\rho}}{\underline{|r}}\right)^2\Sigma_{p=0}^{p=r}\frac{\underline{|n+1}}{\underline{|n-r-\rho+p}\;\underline{|r+\rho-p+1}}\;\frac{\underline{|r}}{\underline{|p}\;\underline{|r-p}}.$$

Now

$$\frac{\underline{|n+1}}{\underline{|n-r-\rho+p}\;\underline{|r+\rho-p+1}} \text{ is } = \text{coeff. of } x^{r+\rho-p+1} \text{ in expansion of } (1+x)^{n+1},$$

and

$$\frac{\underline{|r}}{\underline{|p}\;\underline{|r-p}} \text{ is } = \text{coeff. of } x^p \quad \text{in expansion of } (1+x)^r;$$

therefore

$$\Sigma_{p=0}^{p=r}\frac{\underline{|n+1}\;\underline{|r}}{\underline{|n-r-\rho+p}\;\underline{|r+\rho-p+1}\;\underline{|p}\;\underline{|r-p}}$$

is $=$ coeff. of $x^{r+\rho+1}$ in the expansion of $\overline{1+x}^{n+r+1}$, and is therefore

$$= \frac{\underline{|n+r+1}}{\underline{|r+\rho+1}\;\underline{|n-\rho}};$$

Hence

$$_rB_\rho = \frac{\underline{|n-r}\;\underline{|2r+1}}{\underline{|r+\rho+1}\;\underline{|n-\rho}\;\underline{|\rho-r}}\left(\frac{\rho}{r}\right)^2,$$

$$= \frac{\overline{(n-r)}^{(\rho-r)}}{(r+\rho+1)^{(\rho-r)}}\;\frac{\{\rho^{(\rho-r)}\}^2}{\underline{|\rho-r}}. \qquad \text{(iv)}$$

Now the original equations for finding $_rA_r$, $_rA_{r-1}$, &c. may be written

$$ {}_rA_t\, \Sigma_{x=0}^{x=n}\, x^{(t)}\, x^{(s)} + {}_rA_{t-1}\, \Sigma_{x=0}^{x=n}\, x^{(t-1)}\, x^{(s)} \ldots $$

$$ + {}_rA_0\, \Sigma_{x=0}^{x=n}\, x^{(s)} - \Sigma_{x=0}^{x=n}\, x^{(s)}\left(y_x - {}_rA_r\, x^{(r)} - \ldots - {}_rA_{t+1}\, x^{(t+1)}\right) = 0\,; $$

so that when

$$ {}_rA_r,\ {}_rA_{r-1} \ldots {}_rA_{t+1} $$

have been determined, ${}_rA_t$ may be obtained in the manner that ${}_rA_r$ was, but with

$$ y_x - {}_rA_r\, x^{(r)} - \ldots - {}_rA_{t+1}\, x^{(t+1)} $$

written for y_x.

Hence by (A)

$$ {}_rA_t = \Sigma_{s=0}^{s=t}\,(-1)^{s+t}\frac{\lfloor 2t+1}{(\lfloor t\ \lfloor s)^2}\,\frac{\lfloor t+s}{\lfloor t-s}\,\frac{1}{(n+t+1)^{(s+t+1)}}\times $$

$$ \Sigma_{x=0}^{x=n}\, x^{(s)}\{y_x - {}_rA_r\, x^{(r)} - \ldots - {}_rA_p\, x^{(p)} - \ldots - {}_rA_{t+1}\, x^{(t+1)}\} $$

$$ = {}_tA_t - {}_tB_r\,.\,{}_rA_r - {}_tB_{r-1}\,.\,{}_rA_{r-1} - \ldots - {}_tB_p\,.\,{}_rA_p - \ldots\ {}_tB_{t+1}\,.\,{}_rA_{t+1} $$

$$ = {}_tA_t - \Sigma_{p=t+1}^{p=r}\ {}_tB_p\,.\,{}_rA_p, $$

$$ = {}_tA_t - \Sigma_{p=t+1}^{p=r}\frac{(n-t)^{(p-t)}\{p^{(p-t)}\}^2}{(t+p+1)^{(p-t)}\ \lfloor p-t}\ {}_rA_p. \qquad \text{(v)} $$

Suppose $t = r-1$, then by (v)

$$ {}_rA_{r-1} = {}_{r-1}A_{r-1} - \frac{n-r+1}{2r}\, r^2\, {}_rA_r. $$

Similarly, if in (v) t be put equal to $r-2$, we get

$$ {}_rA_{r-2} = {}_{r-2}A_{r-2} - \frac{n-r+2}{2r-2}\,(r-1)^2\left\{{}_{r-1}A_{r-1} - \frac{n-r+1}{2r}\, r^2\, {}_rA_r\right\} $$

$$ - \frac{(n-r+2)^{(2)}}{(2r-1)^{(2)}}\,\frac{\{r^{(2)}\}^2}{\lfloor 2}\ {}_rA_r, $$

$$ = {}_{r-2}A_{r-2} - \frac{n-r+2}{2(r-1)}\,(r-1)^2\ {}_{r-1}A_{r-1} + \frac{(n-r+2)^{(2)}\{r^{(2)}\}^2}{2\,(2r)^{(2)}}\ {}_rA_r. $$

In the same way if t be put equal to $x-3$ in (v), it will be found that the resulting equation may be reduced to the form

$$ {}_rA_{r-3} = {}_{r-3}A_{r-3} - \frac{n-r+3}{2r-4}\,(r-2)^2\ {}_{r-2}A_{r-2} $$

$$ + \frac{(n-r+3)^{(2)}}{\lfloor 2}\,\frac{\{(r-1)^{(2)}\}^2}{(2r-2)^{(2)}}\ {}_{r-1}A_{r-1} - \frac{(n-r+3)^{(3)}\{r^{(3)}\}^2}{\lfloor 3\ (2r)^{(3)}}\ {}_rA_r. $$

We see then that as far as ${}_rA_{r-3}$ the value of ${}_rA_s$ is of the form

$$\Sigma_{t=s}^{t=r}(-1)^{s+t}\frac{(n-s)^{(t-s)}\{t^{(t-s)}\}^2}{(2t)^{(t-s)}\,\underline{|\,t-s}}\,{}_tA_t. \qquad \text{(B)}$$

Let us assume that all the coefficients ${}_rA_{r-1}$, ${}_rA_{r-2}$ . . . ${}_rA_s$ are of this form; we will prove that ${}_rA_{s-1}$ must also be of this form.

For by (v)

$${}_rA_{s-1} = {}_{s-1}A_{s-1} - \Sigma_{p=s}^{p=r}\frac{(n-s+1)^{(p-s+1)}\{p^{(p-s+1)}\}^2}{(s+p)^{(p-s+1)}\,\underline{|\,p-s+1}}\,{}_rA_p$$

$$= {}_{s-1}A_{s-1} - \Sigma_{p=s}^{p=r}\frac{(n-s+1)^{(p-s+1)}\{p^{(p-s+1)}\}^2}{(s+p)^{(p-s+1)}\,\underline{|\,p-s+1}}\,\Sigma_{t=p}^{t=r}(-1)^{p+t}\frac{(n-p)^{(t-p)}\{t^{(t-p)}\}^2}{(2t)^{(t-p)}\,\underline{|\,t-p}}\,{}_tA_t$$

(for, by hypothesis, B holds for all values of from s to r); therefore

$${}_rA_{s-1} = {}_{s-1}A_{s-1} - \Sigma_{p=s}^{p=r}\,\Sigma_{t=p}^{t=r}(-1)^{p+t}\frac{(n-s+1)^{(t-s+1)}\{t^{(t-s+1)}\}^2}{(2t)^{(t-p)}\,\underline{|\,t-p}\,(s+p)^{(p-s+1)}\,\underline{|\,p-s+1}}\,{}_tA_t;$$

or, changing the order of the summations,

$${}_rA_{s-1} = {}_{s-1}A_{s-1} - \Sigma_{t=s}^{t=r}\,\Sigma_{p=s}^{p=t}(-1)^{p+t}\frac{(n-s+1)^{(t-s+1)}\{t^{(t-s+1)}\}^2}{(2t)^{(t-p)}\,\underline{|\,t-p}\,(s+p)^{(p-s+1)}\,\underline{|\,p-s+1}}\,{}_tA_t.$$

Now

$$\frac{\underline{|\,t-s+1}}{\underline{|\,t-p}\,\underline{|\,p-s+1}} \text{ is } = \text{coeff. of } x^{t-p} \text{ in expansion of } (1+x)^{t-s+1},$$

and

$$(-1)^{p+s}\frac{\underline{|\,t+p}}{\underline{|\,t-s}\,\underline{|\,p+s}} \text{ is } = \text{coeff. of } x^{p+s} \text{ in expansion of } (1+x)^{-t+s-1};$$

Hence

$$\Sigma_{p=s-1}^{p=t}(-1)^{p+s}\frac{\underline{|\,t-s+1}}{\underline{|\,t-p}\,\underline{|\,p-s+1}}\,\frac{\underline{|\,t+p}}{\underline{|\,t-s}\,\underline{|\,p+s}}$$

is equal to the coefficient of x^{t+s} in the expansion of $(1+x)^0$ and is therefore zero. Hence

$$\Sigma_{p=s}^{p=t}(-1)^{p+s}\frac{\underline{|\,t+p}}{\underline{|\,t-p}\,\underline{|\,p-s+1}\,\underline{|\,p+s}} = \frac{\underline{|\,t+s-1}}{\underline{|\,t-s+1}\,\underline{|\,2s-1}};$$

or, finally,

$${}_rA_{s-1} = {}_{s-1}A_{s-1} - \Sigma_{t=s}^{t=r}(-1)^{s+t}\frac{(n-s+1)^{(t-s+1)}\{t^{(t-s+1)}\}^2}{\underline{|\,2t}}\,\underline{|\,2s-1}\,\frac{\underline{|\,t+s-1}}{\underline{|\,t-s+1}\,\underline{|\,2s-1}}\,{}_tA_t,$$

$$= \Sigma_{t=s-1}^{t=r}(-1)^{s+t-1}\frac{(n-s+1)^{(t-s+1)}\{t^{t-s+1}\}^2}{(2t)^{t-s+1}\,\underline{|\,t-s+1}}\,{}_tA_t,$$

which is of the form (B) with $s-1$ in place of s.

But ${}_rA_{r-1}$, ${}_rA_{r-2}$ have been already shown to be of the form (B), so therefore is ${}_rA_{r-3}$, and so then ${}_rA_{r-4}$ &c. Hence, for all values of s from $s=0$ to $s=r-1$, ${}_rA_s$ is of the form given by (B).

The result of the foregoing investigation may be thus stated: If ${}_rA_s$ be the coefficient of $x^{(s)}$ obtained, by the method of least squares, from the $\overline{n+1}$ values of y_x when $x=0, 1, 2, \ldots n$, n being greater than r, and y_x being of the form

$$ {}_rA_r\, x^{(r)} + {}_rA_{r-1}\, x^{(r-1)} + \ldots + {}_rA_s\, x^{(s)} + \ldots + {}_rA_0\,; $$

then

$$ A_s = \Sigma_{t=s}^{t=r} (-1)^{t+s} \frac{(n-s)^{(t-s)} \{t^{(t-s)}\}^2}{(2t)^{(t-s)} \underline{|t-s}}\, {}_tA_t, \tag{B} $$

where

$$ {}_tA_t = \Sigma_{\rho=0}^{\rho=t} (-1)^{\rho+t} \frac{\underline{|2t+1}}{(\underline{|t}\,\underline{|\rho})^2} \frac{(t+\rho)^{(2\rho)}}{(n+t+1)^{(t+\rho+1)}} \Sigma_{x=0}^{x=n} x^{(\rho)}\, y_x. \tag{A} $$

The following are the values of the constants, calculated from these formulæ, as far as they are required if fifth differences are assumed to be constant.

$$ {}_0A_0 = \frac{1}{n+1} \Sigma_{x=0}^{x=n} y_x. $$

$$ {}_1A_1 = \frac{-6}{(n+2)(n+1)} \Sigma_{x=0}^{x=n} y_x + \frac{12}{(n+2)(n+1)\,n} \Sigma_{x=0}^{x=n} x\, y_x. $$

$$ {}_2A_2 = \frac{30}{(n+3)(n+2)(n+1)} \Sigma_{x=0}^{x=n} y_x - \frac{180}{(n+3)(n+2)(n+1)\,n} \Sigma_{x=0}^{x=n} x\, y_x + \frac{180}{(n+3)(n+2)\ldots(n-1)} \Sigma_{x=0}^{x=n} x(x-1)\, y_x. $$

$$ {}_3A_3 = \frac{-140}{(n+4)(n+3)\ldots(n+1)} \Sigma_{x=0}^{x=n} y_x + \frac{1680}{(n+4)(n+3)\ldots n} \Sigma_{x=0}^{x=n} x\, y_x - \frac{4200}{(n+4)(n+3)\ldots(n-1)} \Sigma_{x=0}^{x=n} x(x-1)\, y_x $$
$$ + \frac{2800}{(n+4)(n+3)\ldots(n-2)} \Sigma_{x=0}^{x=n} x(x-1)(x-2)\, y_x. $$

$$ {}_4A_4 = \frac{630}{(n+5)(n+4)\ldots(n+1)} \Sigma_{x=0}^{x=n} y_x - \frac{12600}{(n+5)(n+4)\ldots n} \Sigma_{x=0}^{x=n} x\, y_x + \frac{56700}{(n+5)(n+4)\ldots(n-1)} \Sigma_{x=0}^{x=n} x(x-1)\, y_x $$
$$ - \frac{88200}{(n+5)(n+4)\ldots(n-2)} \Sigma_{x=0}^{x=n} x(x-1)(x-2)\, y_x + \frac{44100}{(n+5)(n+4)\ldots(n-3)} \Sigma_{x=0}^{x=n} x(x-1)(x-2)(x-3)\, y_x. $$

$$ {}_5A_5 = \frac{-2772}{(n+6)(n+5)\ldots(n+1)} \Sigma_{x=0}^{x=n} y_x + \frac{83160}{(n+6)(n+5)\ldots n} \Sigma_{x=0}^{x=n} x\, y_x - \frac{582120}{(n+6)(n+5)\ldots(n-1)} \Sigma_{x=0}^{x=n} x(x-1)\, y_x $$
$$ + \frac{1552320}{(n+6)(n+5)\ldots(n-2)} \Sigma_{x=0}^{x=n} x(x-1)(x-2)\, y_x - \frac{1746360}{(n+6)(n+5)\ldots(n-3)} \Sigma_{x=0}^{x=n} x(x-1)(x-2)(x-3)\, y_x $$
$$ + \frac{698544}{(n+6)(n+5)\ldots(n-4)} \Sigma_{x=0}^{x=n} x(x-1)(x-2)(x-3)(x-4)\, y_x. $$

$$
\begin{aligned}
{}_rA_0 &= {}_0A_0 - \frac{n}{2}\,{}_1A_1 + \frac{n\,(n-1)}{6}\,{}_2A_2 - \frac{n\,(n-1)\,(n-2)}{20}\,{}_3A_3 + \frac{n\,(n-1)\,(n-2)\,(n-3)}{70}\,{}_4A_4 - \frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)}{252}\,{}_5A_5 + \&c.\\
{}_rA_1 &= {}_1A_1 - (n-1)\,{}_2A_2 + \frac{3}{5}(n-1)\,(n-2)\,{}_3A_3 - \frac{2}{7}(n-1)\,(n-2)\,(n-3)\,{}_4A_4 + \frac{5}{42}(n-1)\,(n-2)\,(n-3)\,(n-4)\,{}_5A_5 - \&c.\\
{}_rA_2 &= {}_2A_2 - \frac{3}{2}(n-2)\,{}_3A_3 + \frac{9}{7}(n-2)\,(n-3)\,{}_4A_4 - \frac{5}{6}(n-2)\,(n-3)\,(n-4)\,{}_5A_5 + \&c.\\
{}_rA_3 &= {}_3A_3 - 2\,(n-3)\,{}_4A_4 + \frac{20}{9}(n-3)\,(n-4)\,{}_5A_5 - \&c.\\
{}_rA_4 &= {}_4A_4 - \frac{5}{2}(n-4)\,{}_5A_5 + \&c.
\end{aligned}
$$

If fourth differences are assumed constant, ${}_5A_5$ must be put equal to zero; if third differences are taken as constant, ${}_4A_4$ and ${}_5A_5$ must be put equal to zero; and so on.

In employing these formulæ, it will generally save labour if we subtract from each of the quantities y_x the quantity given by the formula

$$C_r x^{(r)} + C_{r-1} x^{(r-1)} + \ldots + C_s x^{(s)} + \ldots + C_0,$$

where $C_r, C_{r-1}, \ldots C_0$ are so chosen as to make

$$y_x - (C_r x^{(r)} + C_{r-1} x^{(r-1)} + \ldots + C_0) = 0$$

for all values of x from $x = n - r$ to $x = n$. Thus all the values of y_x for which the coefficient is large will vanish, and the other values will also, as a rule, be very much diminished. The formulæ (A) and (B) may then be employed, using the remainders, after subtraction, instead of the original values of y_x; and if ${}_rA'_s$ is one of the coefficients so determined, the coefficient in the original value of y_x will be

$$C_s + {}_rA'_s.$$

The above investigation may also be modified, so as to apply to the case when the coefficients in the value for y_x are known to be connected by one or more linear relations, and also for approximating to the values when the relations are not linear.

For, suppose it known that

$$a_r \cdot {}_rA_r + a_{r-1} \cdot {}_rA_{r-1} + \ldots + a_\rho \cdot {}_rA_\rho + \ldots + a_0 \cdot {}_rA_0 + q = 0, \qquad \text{(vi)}$$

we must make

$$\Sigma_{x=0}^{x=n} ({}_rA_r x^{(r)} + {}_rA_{r-1} x^{(r-1)} + \ldots + {}_rA_\rho x^{(\rho)} + \ldots + {}_rA_0 - y_x)^2$$
$$- \lambda (a_r \cdot {}_rA_r + a_{r-1} \cdot {}_rA_{r-1} + \ldots + a_\rho \cdot {}_rA_\rho + \ldots + a_0 \cdot {}_rA_0 + q)$$

a minimum.

The resulting equations are of the same form as before, except that now

$$\lambda a_\rho + \Sigma_{x=0}^{x=n} x^{(\rho)} y_x$$

occurs instead of

$$\Sigma_{x=0}^{x=n} x^{(\rho)} y_x,$$

ρ having any value from 0 to r.

Replacing

$$\Sigma_{x=0}^{x=n} x^\rho y_x \text{ by } \lambda a_\rho + \Sigma_{x=0}^{x=n} x^{(\rho)} y_x$$

in the solution, we get

$$ {}_tA_t = \Sigma_{\rho=0}^{\rho=t} (-1)^{\rho+t} \frac{\lfloor 2t+1}{(\lfloor t \lfloor \rho)^2} \frac{(t+\rho)^{(2\rho)}}{(n+t+1)^{(t+\rho+1)}} \{\lambda a_\rho + \Sigma_{x=0}^{x=n} x^{(\rho)} y_x\}; \qquad \text{(vii)} $$

whence ${}_rA_s$ &c. may be determined in terms of λ. The value of λ must then be determined, so that (vi) is satisfied; and this value must then be substituted in the expressions for ${}_rA_s$, &c.

If the relation be not linear, suppose ϕ (${}_rA_r$, ${}_rA_{r-1}$, &c.) be the relation.

Let ${}_rA'_r$, ${}_rA'_{r-1}$, &c. be the values of the constants found by means of (A) and (B) without reference to the given relation. Suppose ${}_rA_r = {}_rA'_r + \delta\, {}_rA'_r$ &c., then

$$ \phi\, ({}_rA'_r + \delta\, {}_rA'_r,\ {}_rA'_{r-1} + \delta\, {}_rA'_{r-1},\ \&c.) = 0; $$

therefore

$$ \phi\, ({}_rA'_r,\ {}_rA'_{r-1},\ \&c.) + \delta\, {}_rA'_r \frac{d\phi}{d\, {}_rA'_r} + \delta\, {}_rA'_{r-1} \frac{d\phi}{d\, {}_rA'_{r-1}} + \&c. = 0, $$

or

$$ {}_rA_r \frac{d\phi}{d\, {}_rA'_r} + {}_rA_{r-1} \frac{d\phi}{d\, {}_rA'_{r-1}} + \ldots + \phi\, ({}_rA'_r,\ {}_rA'_{r-1},\ \&c.) - {}_rA'_r \frac{d\phi}{d_r A'_r} - {}_rA'_{r-1} \frac{d\phi}{d\, {}_rA'_{r-1}} - \&c. = 0. $$

If we suppose that $\delta\, {}_rA_r$ &c. are so small that their squares and products may be neglected, then $\dfrac{d\phi}{d_r A'_r}$ &c. may be treated as constants; the relation will now be of the form (vi) and we may find as above the values of ${}_rA_r$, ${}_rA_{r-1}$, &c.

Meteorological Office, Toronto, Canada,
1879, *April* 30.

Royal Observatory, Greenwich,
1879, *July* 29.

It is known generally to the Society that meridional observations on a comprehensive scale are carried on at the Royal Observatory with unceasing regularity; but it is not so generally known that a considerable number of extrameridional or irregular observations have accumulated during my superintendence of the Observatory.

I have thought, therefore, that I might with propriety lay before the Society an arranged list of observations of the last-mentioned classes, nearly the whole of which are included in the annual volumes of *Greenwich Observations.*

G. B. Airy.